1D Runoff Production in the Light of Queueing Theory: Heterogeneity, Connectivity and Scale

Marie-Alice Harel[1], Emmanuel Mouche[2]

Abstract Runoff production on a hillslope during a rainfall event is governed by different processes, among which the well-known runoff-runon process. It arises at different scales, on plots or hillslopes which display heterogeneous infiltration rates. The runoff produced in an area of low infiltration rate may re-infiltrate downslope in an area of high infiltration rate. Therefore runoff is organized in patterns of random position and extension. When the rainfall rate increases these zones connect and produce the runoff peak as observed in streams during rainfall events, for instance. The 1D stationary runoff-runon equations are similar to the queueing equations in probability and statistics. In this framework we investigate by means of numerical simulation the influence of the infiltration rate statistics on the runoff formation. We show that the statistics greatly influence the extension of the runoff patterns and the associated flow rates.

[1] Climate and Environment Sciences Laboratory, Centre d'Etudes de Saclay, Orme des Merisiers, 91191 Gif sur Yvette, France. mharel@lsce.ipsl.fr

[2] Climate and Environment Sciences Laboratory, Centre d'Etudes de Saclay, Orme des Merisiers, 91191 Gif sur Yvette, France. emmanuel.mouche@cea.fr

1. Introduction

Assessment and prediction of peak flow genesis or water erosion during a rainfall event involves an accurate knowledge of runoff formation, runoff volume and rate during the event. Indeed, these physical characteristics depend on the rainfall intensity evolution during the event, land-cover as well as many soil parameters such as soil composition, moisture content, crusting and slope.

At the plot or hillslope scale, runoff is often generated by infiltration excess: when a rainfall event starts, soil surface is quickly saturated by rainfall infiltration. After surface saturation, rainfall that cannot infiltrate ponds and produce runoff if the slope gradient is sufficient. This process was described originally by Horton [4] who proposed an empirical expression for the time evolution of the infiltration rate. Horton's description is justified for ideal plots (homogeneous soil, plane surface, no vegetation and no surface evolution) or homogeneous soil columns used in laboratory infiltration experiments. Horton states that when the rainfall rate, assumed to be constant with time, is greater than the soil hydraulic conductivity the infiltration rate tends over time to the hydraulic conductivity, whatever the rainfall rate value. Observation of field infiltration at the plot-scale, for instance, under natural or simulated rainfall, often exhibits a dependency upon rainfall intensity, which is inconsistent with Horton's description. Indeed, simulated rainfall experiments show that steady state infiltration rate increases with rainfall intensity, and may tend to a plateau at high rainfall intensities.

Two reasons are put forth by authors to explain qualitatively this dependency: Point infiltration rate variability and microtopography (see for instance the review of Dunne [3]). At the plot scale crusts, vegetation cover, micro topography and their combined effects generate a heterogeneous soil surface with a wide panel of infiltration rates. Therefore the point infiltration rate varies spatially and the fraction of the plot area for which hydraulic conductivity is smaller than the rainfall rate increases with this rate. As a result the apparent infiltration rate, sum of the point infiltration rates divided by the plot area, is an increasing function of the rainfall rate. Concerning the impact of microtopography it has been observed that some of the rainfall water is stored in microdepressions and when rainfall rate increases this stored water may be released and routed downslope to others micro-pools where it may infiltrate or stored again, or to the plot outlet producing measurable runoff, see for instance [2] for a discussion and a simulation of the process. Moreover, infiltration variability and microtopography give rise to the well-known runoff-runon process where runoff generated upslope in a low permeability area may infiltrate downslope in a permeable area [8,12].

The complexity of the processes involved in runoff production leads many authors to use empirical approaches to predict runoff and its increase with rainfall intensity. These are essentially the Soil Conservation Service (SCS) runoff curve number or the even simpler runoff coefficient approach. This partitioning coefficient, like the SCS equation, cannot be linked by a model to the soil physical properties. Since the end of the seventies different authors tried to propose alternative solutions based on simple physical considerations on the infiltration rate variability on a plot.

Lafforgue [10] is one of the first authors who investigated theoretically the role of this variability on runoff production. Based on field observations made in West-Africa he proposed to model the plot as a statistical ensemble of point infiltration rates described either by a uniform or a bimodal law. Within this approach he showed how the mean infiltration rate over the plot increases with the rainfall rate. Hawkins [7,8] proposed also the same statistical approach assuming that the plot may be viewed as a set of parallel strips aligned along the slope with infiltration rates given by an exponential distribution. Hawkins showed that infiltration rate increases exponentially with rainfall rate and tends to a maximum value equal to the mean statistical infiltration rate and called the maximum infiltration rate. The opposite case where the strips are orthogonal to the slope has never been investigated theoretically. Indeed, it raises the problem of how to take into account the runoff-runon process in the runoff transport equation.

The aim of this paper is to show how queueing theory, developed in operational research for telecommunications and others fields, may serve, as already observed by Jones [9], as a framework to take into account the runon-runoff process. We consider the case of a constant rain falling on a slope made up of pixels with infiltration rate distributed randomly according to different laws. The influence of the infiltration rate statistics on the runoff formation is analysed.

2. Runon-runoff as a queueing process

The domain, which may represent either a plot or a hillslope, is modelled as a simple idealized one dimensional flat soil surface. Microtopography, and consequently runoff storage in microdepressions, is neglected. Local infiltration is described by a threshold infiltration function depending on a single parameter called soil infiltrability. Infiltrability encompasses all physical processes that influence infiltration locally, such as crusting, vegetation or soil structure. It is the maximal rate of water that the soil can infiltrate. In this approach the ponding time is neglected and all the available water (runon plus rainfall) infiltrates instantly as long as water supply is inferior to infiltrability. Infiltration excess overland flow is therefore the only mechanism supposed to produce runoff.

The one dimensional domain is divided into N pixels of size Δx. The infiltrability I is assumed to be constant over time and distributed randomly in each pixel. The rainfall rate is also independent of time and set uniformly along the slope. Let us call Q_i the runoff flow per unit width from pixel i to pixel $i+1$, where index i increases down slope. At steady state the runoff mass balance in pixel i is given by:

$$Q_{i+1} = Max(Q_i + R - I_i\ ,\ 0) \tag{1}$$

where I_i is the infiltrability in pixel i. If $Q_i = 0$ then runoff is produced in pixel $i+1$ only for $R > I_{i+1}$ and if $Q_i \neq 0$ runoff is produced as long as $Q_i + R > I_{i+1}$ (Fig. 1). The total water flow rate $Q_i + R$ can be considered as an effective rainfall rate for pixel i. The boundary condition in pixel $i = 1$ is an imposed runoff flow, which may be zero (at the top of a hillslope) or non-zero (at the top of a plot for instance).

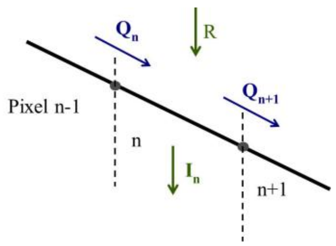


Fig. 1 Notations for the slope discretization in pixels. Q_{n+1} is the runoff flowing out of pixel n onto pixel $n+1$. It is the result of runoff mass balance over pixel n, where I_n is the infiltrability, R rainfall amount falling over that pixel and Q_n the upslope runoff input (*i.e.* runon).

The above runon-runoff equation (1) describes a random nonlinear sequence which cannot be approximated by a more tractable equation. Simulation shows that, for a realization of a given infiltrability random field, the corresponding random runoff flow rate field defines wet ($Q \neq 0$) and dry ($Q = 0$) areas of random positions and extensions in the domain. Indeed, the areas producing runoff, where $R > I_i$, are included in the wet areas and the difference between the wet area and the area producing runoff is due to runon occurrence.

Jones et al. [9] were the first authors to remark that the runoff-runon equation (1) is identical to the waiting time equation in a single server queue. These authors used the considerable mathematical literature on queueing systems to derive some statistical results on the runoff flow rate (mean and variance) and on the connected length at the bottom of a domain. They assumed no spatial correlation of infiltrability between two pixels (white noise random field). In queueing theory Q_i is the waiting time of customer i, R the service time and I_i the inter-arrival time between customers i and i-1. The traffic parameter ρ, defined by equation (2), stands for the adimensioned rainfall intensity in hydrology. For an introduction to the queueing theory, refer to [11] for instance.

$$\rho = R \,/\, \langle I \rangle \tag{2}$$

Thanks to the queueing theory framework, equation (1) may be solved for some infiltrability distributions like the classical exponential white noise. In this work we consider three others distributions: uniform, bimodal and log-normal. The basic quantity to compute is the distribution of the runoff flow rate Q, $Pr(Q \leq x)$, which leads to different quantities of interest: the mean and the variance of the flow rate, the wet area fraction, $1\text{-}Pr(Q = 0)$, the dry area fraction, $Pr(Q = 0)$, the distribution of the runoff patterns lengths, the mean number of these patterns per unit length, as well as connectivity parameters. All these quantities are further discussed in [5], where we also propose some theoretical results for the bimodal distribution that will be used in the following. The two other distributions have no solution.

3. Simulations and results

We consider a flat 1D slope of length $L = 12\ 000$ pixels, where each pixel is assigned a value of infiltrability generated according to an uncorrelated probability distribution chosen among log-normal, exponential, bimodal and uniform. The log-normal distribution is characteristic of soil infiltrability over basin scales, whereas the exponential is commonly employed to represent smaller scales such as plots and bare soils. The bimodal distribution may be useful to generate synthetic crusted soils: two surface states or two different soils, one with high infiltration capacity, the other non-infiltrating (the proportion of each of these values is set at 50% in this study). Finally, the uniform distribution was considered rather for its mathematical properties and simplicity. This comprehension of distributions as representations of specific scales or soil characteristics may be exploited when analysing the following results, thus improving our understanding of runoff scale effect (more in [5]).

The random variable is generated with software R functions for random numbers. The rainfall intensity imposed uniformly over the slope varies between zero and two times the mean infiltrability ($0 \leq \rho \leq 2$). In order to observe a stationary runoff distribution for rainfall values up to $\rho = 0.9$ approximately, the 2000 first upslope pixels are removed from all simulations. Consequently the real domain length is 10 000 pixels long.

Figure 2) displays the distribution of the runoff flow rate along the slope for the four infiltrability distributions at $\rho = 0.4$. We observe clearly that the spatial organization of runoff depends on the nature of the infiltrability distribution. The figure also shows the runoff-runon process: runoff occurs in zones where $R > I$. The highest runoff flow occurs for the bimodal distribution and the lowest for the uniform (the exponential is second highest after the bimodal). This hierarchy can be easily understood as for an identical rainfall rate, each distribution intercept the threshold R differently. The runoff production is even more enhanced with increased proportion of infiltrability below R. In this respect, the bimodal distribution is extreme as all low infiltrability values are set to zero.

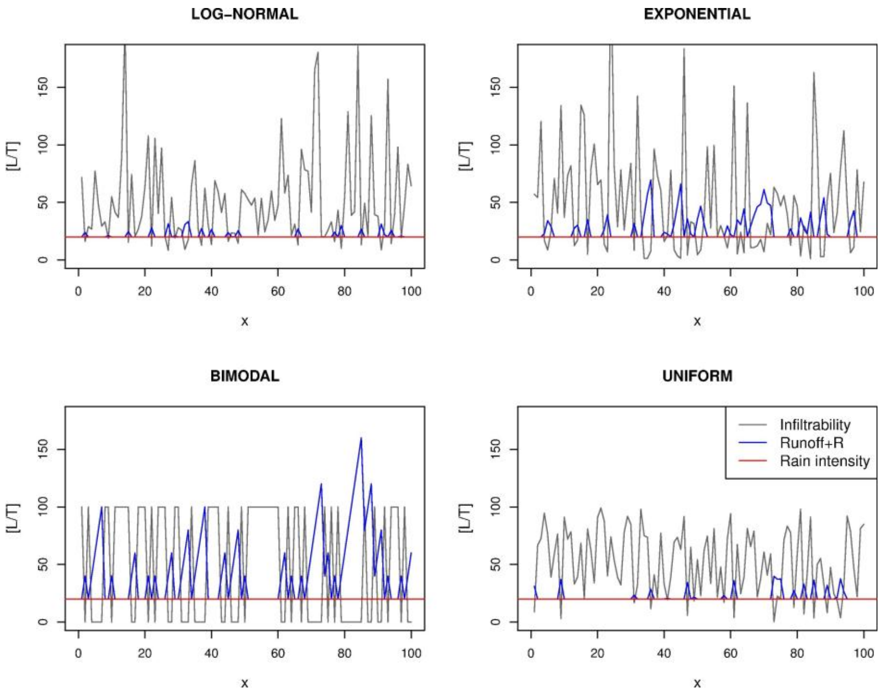
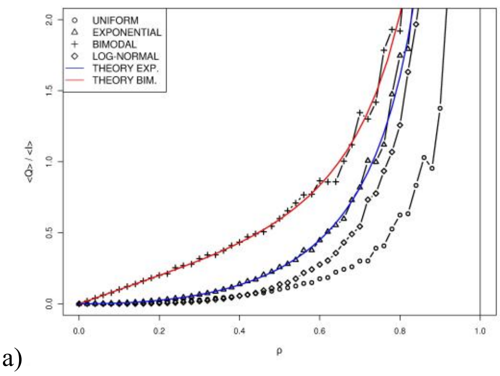
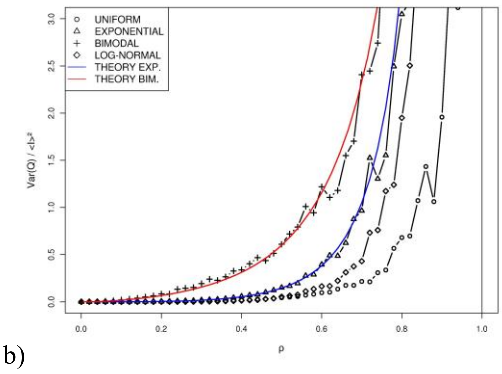


Fig. 2 Spatial organisation of runoff (in blue) and infiltrability (grey) over the slope for the different infiltrability distributions, at $\rho = 0.4$ (rainfall intensity in green).

Figures 3 a) and b) display respectively the mean and the variance of the runoff flow rate, adimensioned by the mean infiltrability, as functions of ρ and for the four infiltrability distributions. The nonlinearity of runoff flow rate as a function of rainfall intensity is obvious, especially for values of ρ lying between 0.5 and 1. For $\rho > 1$, runoff becomes linear and is no more influenced by the infiltrability distribution, as the entire slope is overflown. For $\rho < 1$, the infiltrability distribution has a great influence on the runoff production. For a given rainfall intensity, the mean flow rate is multiplied by several orders of magnitude between bimodal and uniform infiltrability distributions. This follows the same ranking as the one discussed previously. High mean runoff means extremes of infiltrability in the low values. Figure 3b) shows that the high runoff production relates to high runoff variability and the hierarchy between distributions observed in term of mean runoff is conserved.



a)



b)

Fig. 3 Adimensioned mean runoff rate (a) and runoff variance (b), as a function of rainfall rate (ρ). The theoretical results obtained from the queueing theory are displayed as solid lines (blue for the exponential distribution, red for bimodal) and compared to the numerical simulations (symbol lines).

Figure 4) compares the wet area fractions, that is the fraction of pixels on which runoff occurs, for the different infiltrability distributions. The wet area fraction for the exponential distribution is linear, as predicted by theory (blue line), and is equal to ρ. The log-normal infiltrability evolve somehow in the same way, the fraction is lower for small values of ρ and reaches unity almost linearly. However, for rainfall intensities below 0.5, the log-normal density is much lower than the exponential. The exponential probability density function (pdf) has its highest probability at infiltrability zero, whereas the log-normal pdf is maximal for low but non-zero values of infiltrability. This is why the density of runoff producing pixels is higher for exponential pdf than for log-normal ones: there is a high probability that runoff is readily produced for the smallest non-null rainfall in the exponential case. The uniform density curve is the lowest, except for reduced rainfall intensities, which observation supports the associated low mean runoff values discussed in figure 3).

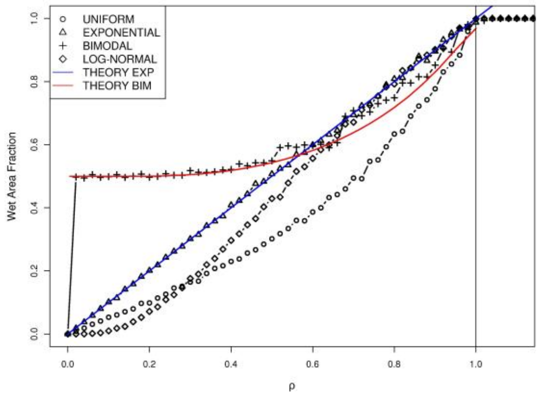


Fig. 4 Runoff wet area fraction (number of pixels where runoff occurs over the total number of pixels) compared between the four infiltrability distributions and confronted to the theory (colour lines).

Finally, it is the bimodal wet area fraction that is most strong-shaped and original. Our theoretical developments (red line) accurately predict the observed evolution. The curve starts at the origin without rainfall, then 50% of the domain is immediately flooded for the smallest rainfall intensity. Then the fraction is almost constant until $\rho = 0.5$ approximately, and it increases linearly for rainfall amounts above $\rho = 0.6$. The wet area fraction does not increase notably for the low rainfall intensities because there is a threshold at $\rho = 0.5$ that need to be breached. The bimodal slope being a random succession of pixels of either *0* or $2\langle I \rangle$ infiltrability values, only a rare series of non-infiltrating pixels can generate a sufficient runoff amount to overcome the threshold of $2\langle I \rangle$ at the end of the series. On the other hand, as soon as rainfall intensity exceeds $\rho = 0.5$, even a single non-infiltrating pixel flanked by two infiltrating pixels will overflow the next downslope element. Consequently, the runon process is obviously non-active for small rainfall intensities (which is true for any distribution to a certain degree, as commented just then) and the wet area fraction for the bimodal case increases very slowly under the threshold.

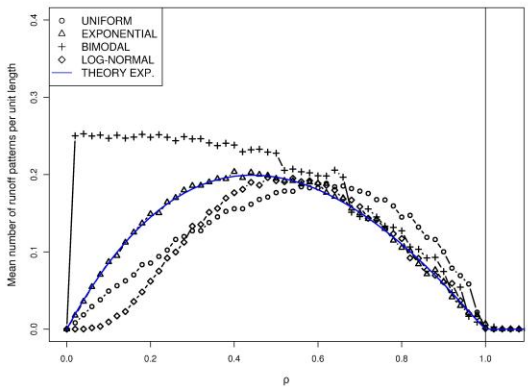


Fig. 5 Mean number of runoff patterns per unit length for each distribution, with the exponential theory in blue, as a function of the adimensioned rainfall rate.

The mean number of runoff patterns per unit length, as shown in figure 5), confirms the results already discussed over figure 4), but also brings some more precisions about the organization of the flow. At low rainfall intensities, the hierarchy bimodal > exponential > uniform > log-normal mirrors the last discussion about the wet area fraction. Again we find that the mean number of patterns for the bimodal distribution is almost constant until $\rho = 0.5$. Thus the slope is covered by numerous small runoff patterns that begin to widen and connect to each other when the threshold is breached. On the contrary, for the other distributions of infiltrability, the maximum amount of runoff patterns is encountered for rainfall intensities around $\rho = 0.6$. For these distributions, runoff organizes in rather small size and isolated runoff patterns for low values of ρ. As the rainfall amount increases, more and more patterns appear until the runon process is strong enough to connect neighboring patterns. When the rainfall reaches this critical value, which correspond to the dominance of runon and connectivity over pattern generation, the mean number of runoff patterns decrease. The runon process widens and connects all the patterns, rather quickly for the bimodal distribution and slowly for the uniform, until the slope is overflown with one last runoff pattern. The critical rainfall is different for each distribution of soil infiltrability. It is high for the uniform and low for the exponential. This means, as hinted previously, that the exponential distribution rather favours the runon process and the resulting aggregation of runoff clusters, whereas uniform distributions prevent it to some extent. This remark is consistent with the mean runoff hierarchy in figure 3), that is higher runoff amounts leads to higher patterns appearance, thus higher probability of merging patterns and improved connectivity.

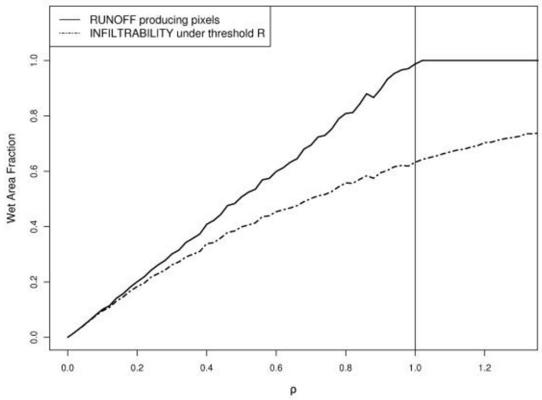


Fig. 6 For the exponential distribution, comparison of the runoff wet area fraction (solid line) with the fraction of pixels for which the rainfall rate exceeds infiltrability (dotted line), as a function of rainfall.

In figure 6), we compare the wet area fraction defined for the flow rate Q (same as in fig. 4) with the fraction of pixels for which $I < R$ along the slope for the exponential distribution, as a function of rainfall rate. If the runoff-runon process was not modelled, equation 1) would be linear and runoff would only be produced where rainfall exceeds the soil infiltrability. Therefore the difference between the solid and dotted line reflects the evolution of runon importance with ρ. We understand that while the rainfall rate is small ($\rho < 0.2$ approximately), runon scarcely occurs and both curves are identical. Then as ρ increases, the runoff production becomes non-linear because of the runon process occurrence. This is why the wet area fraction always exceeds and differs from the fraction of pixels where runoff is produced ($I < R$).

4. Conclusion

We showed in this paper that queueing theory is well suited for the description of runoff formation, especially the runon-runoff process. The connectivity of the runoff layer, as defined by Allard [1], is discussed in [5]. We also underlined the strong link between infiltrability and runoff statistics: the bimodal infiltrability distribution leads to the highest runoff flow rates, whereas the uniform to the lowest. The next step is the study of the transient runoff formation during a rainfall event [6]. For this purpose the queueing theory seems to be still the good framework.

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